A B S T R A C T

Quasi-symmetric Domains and j-algebras

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We want to characterize the quasi-symmetric domains among the bounded homogeneous domains.To do that, we use the j-algebraic description of bounded homogeneous domains given in the book by Pyatetskii-Shapiro. This book gives the realization of such a domain as a Siegel domain, constructed by means a so-called j-algebra of the bounded homogeneous domain. We find necessary and sufficient conditions for the cone of the Siegel domain to be self-dual and for Satake's quasi-symmetry condition to be satisfied. Self-duality of the cone occurs when a certain algebra is a Jordan algebra. This j-algebraic characterization is the result of an attempt to give a geometric characterization of quasi-symmetric domains.

§1. Introduction.

According to Pyatetskii-Shapiro ([9]), if $\mathcal{D}$ is a bounded homogeneous domain, then there is a solvable group which acts simply transitively on $\mathcal{D}$. If $\mathfrak{g}$ is the Lie algebra of this group, then $\mathfrak{g} = \mathfrak{l} + j\mathfrak{l} + \mathfrak{u}$, where j is the pull-back to of the complex structure on $\mathcal{D}$, $\mathfrak{l}$ is an abelian ideal of $\mathfrak{g}$, $j\mathfrak{l}$ is a subalgebra of $\mathfrak{g}$, $[\mathfrak{u},\mathfrak{u}] \subset \mathfrak{l}$, $[j\mathfrak{l},\mathfrak{u}] \subset \mathfrak{u}$ and $[\mathfrak{l},\mathfrak{u}] = 0$. The algebra $\mathfrak{g}$ is a <u>normal j-algebra</u> in the sense that:

(I) $\quad \operatorname{ad} X : \mathfrak{g} \to \mathfrak{g}$ has only real characteristic roots for all $X \in \mathfrak{g}$.

(II) $\quad$ The endomorphism j satisfies $j^2 = -\mathrm{Id}$ and

$$[X,Y] + j[jX,Y] + j[X,jY] - [jX,jY] = 0 \quad \forall X,Y \in \mathfrak{g}.$$

(III) $\quad$ There exists a linear form ω on $\mathfrak{g}$ such that

$$\omega([jX,X]) > 0 \text{ if } X \neq 0 \text{ and } \omega([jX,jY]) = \omega([X,Y]).$$

Remarks: 1) The identity in (II) is the standard integrability condition for the complex structure.

2) Koszul showed that we can find a form ω such that $\omega([X,Y]) = \operatorname{Im} h(X,Y)$, where h is the Bergman metric on $\mathcal{D}$ (pulled back from $\mathcal{D}$ to $\mathfrak{g} \simeq T_0\mathcal{D}$ after choosing a base point 0 in $\mathcal{D}$). (See [9], [7])

By (III) we get the real positive definite inner product $\langle X,Y \rangle = \omega([jX,Y]) = \operatorname{Re} h(X,Y)$ on $\mathfrak{g}$. (See Remark 2)

As shown by Pyatetskii-Shapiro, we have

$$\mathfrak{g} = \mathfrak{h} + \sum_\alpha \mathfrak{k}_\alpha, \text{ vector space direct sum,}$$

where $\mathfrak{h} = [\mathfrak{g},\mathfrak{g}]^\perp$, the orthogonal complement of $[\mathfrak{g},\mathfrak{g}]$, is an abelian subalgebra, $[\mathfrak{g},\mathfrak{g}] = \Sigma \mathfrak{k}_\alpha$ with $\mathfrak{k}_\alpha = \{X \in [\mathfrak{g},\mathfrak{g}] \mid [H,X] = \alpha(H)X \ \forall H \in \mathfrak{h}\}$, where the <u>root</u> α is a linear form on $\mathfrak{h}$. He also

shows that if $\alpha_1,\ldots,\alpha_p$ are all the roots α such that $j\mathfrak{k}_\alpha \subset \mathfrak{h}$, then $\mathfrak{h} = j\mathfrak{k}_{\alpha_1}+\cdots+j\mathfrak{k}_{\alpha_p}$ and $\dim\mathfrak{h} = p$, and further that all roots are of the form α_k , $\frac{1}{2}\alpha_k$ with $1 \leq k \leq p$, $\frac{1}{2}(\alpha_k \pm \alpha_m)$ with $1 \leq k < m \leq p$. We have
We have $j\mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_m)} = \mathfrak{k}_{\frac{1}{2}(\alpha_k-\alpha_m)}$ and $j\mathfrak{k}_{\frac{1}{2}\alpha_k} = \mathfrak{k}_{\frac{1}{2}\alpha_k}$.

If we put $\mathfrak{l} := \sum_{k=1}^{p} \mathfrak{k}_{\alpha_k} + \sum_{1\leq k<m\leq p} \mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_m)}$ and $\mathfrak{U} := \sum_{k=1}^{p} \mathfrak{k}_{\frac{1}{2}\alpha_k}$,

then we get the above mentioned decomposition. It is easy to see that $[\mathfrak{k}_\alpha,\mathfrak{k}_\beta] \subset \mathfrak{k}_{\alpha+\beta}$ and that $\mathfrak{k}_\alpha \perp \mathfrak{k}_\beta$ for $\alpha \neq \beta$. Also $\dim \mathfrak{k}_{\alpha_k} = 1$, and there is an element $E_k \in \mathfrak{k}_{\alpha_k}$ such that $[jE_k,E_k] = E_k$. We let also $E := E_1+\cdots+E_p$. The adjoint representation of the subalgebra $j\mathfrak{l}$ on the ideal $\mathfrak{l}$ gives a corresponding representation of the simply connected group $\mathcal{G}_0$ whose Lie algebra is $j\mathfrak{l}$. Then ([9]) $\Omega := \mathcal{G}_0 \cdot E$ is an open, convex cone in $\mathfrak{l}$ with vertex at the origin, and not containing a whole straight line. By construction Ω is homogeneous, i.e. $Gl(\Omega) := \{ g \in Gl(\mathfrak{l}) \mid g\Omega = \Omega\}$ is transitive on Ω .

The space $\mathfrak{U}$ is a complex vector space with complex structure j , and we define the hermitian form $F : \mathfrak{U} \times \mathfrak{U} \to \mathfrak{l}_{\mathbb{C}}$ by

$$F(u,v) := \frac{1}{4}[ju,v] + \frac{1}{4}i[u,v] ,$$

where $i = \sqrt{-1}$. (Although $\mathfrak{l}_{\mathbb{C}}$ is isomorphic to $\mathfrak{l} + j\mathfrak{l}$, we don't identify, in order to avoid confusions.)
Then $\mathcal{D}(\Omega,F) := \{(z,u) \in \mathfrak{l}_{\mathbb{C}}\times\mathfrak{U} \mid \operatorname{Im} z - F(u,u) \in \Omega\}$ is a Siegel domain ([9]). (Thus $F(u,u) \in \bar{\Omega} \setminus \{0\}$ for $u \neq 0$.) If $\mathcal{G}$ is the simply connected group with Lie algebra $\mathfrak{g}$, then ([9]) there is a representation $\phi : \mathcal{G} \to \mathrm{Aff}_{\mathbb{C}}(\mathfrak{l}_{\mathbb{C}} \times \mathfrak{U})$ with a commutative diagram

$$\begin{array}{ccc} \mathfrak{g} = \mathfrak{l} + j\mathfrak{l} + \mathfrak{u} & \xrightarrow{\phi} & \mathfrak{aff}(\mathfrak{l}_{\mathbb{C}} \times \mathfrak{u}) \\ \exp \downarrow & & \downarrow \exp \\ \mathcal{G} & \xrightarrow{\phi} & \mathrm{Aff}(\mathfrak{l}_{\mathbb{C}} \times \mathfrak{u}) \end{array} ,$$

where the upper ϕ is $\phi(X+jY+W)(z,u) := ([jY,z] + \frac{1}{2}[W,u] - \frac{i}{2}[W,ju] + X, [jY,u] + W)$.

We have for instance

$$\exp jY : (z,y) \to (e^{\mathrm{ad}\,jY}\cdot z,\ e^{\mathrm{ad}\,jY}\cdot u) \quad \text{for} \quad y \in \mathfrak{l}$$

$$\exp(X+W) : (z,u) \to (z + 2iF(u,W) + X + iF(W,W),\ u + W)$$

$$\text{for} \quad X \in \mathfrak{l}\ ,\ W \in \mathfrak{u}\ .$$

It is not difficult to see that $\mathcal{G}$ is simply transitive on $\mathcal{D}(\Omega,F)$, and that, since $\mathcal{G}$ is also simply transitive on $\mathcal{D}$, $\mathcal{D}$ is biholomorphic to $\mathcal{D}(\Omega,F)$.

We want to find out when $\mathcal{D}(\Omega,F) =: \mathcal{D}$ is <u>quasi-symmetric</u> (see below for definition). Since quasi-symmetry involves a self-adjoint cone, we first investigate self-adjointness with respect to $<,>$, i.e. we try to find out when $\Omega^* = \Omega$, where $\Omega^* := \{Y \in \mathfrak{l} \mid <Y,X> > 0\ \forall X \in \bar{\Omega} \setminus \{0\}\}$. We want to use the result of Vinberg about the connection between such cones and Jordan algebras ([11]). For that we need a <u>base point</u> $\xi \in \Omega$ of the following type: If $\mathfrak{gl}(\Omega)$ is the Lie algebra of $Gl(\Omega) := \{g \in Gl(\mathfrak{l}) \mid g\Omega = \Omega\}$, then $\mathfrak{gl}(\Omega) = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of $\{g \in Gl(\Omega) \mid g\xi = \xi\}$, and with $\mathfrak{k} = \{X \in \mathfrak{gl}(\Omega) \mid X' = -X\}$ and $p = \{X \in \mathfrak{gl}(\Omega) \mid X' = X\}$, ' denoting the transpose with respect to $<,>$. We have $\mathrm{ad}\, j\mathfrak{l} \subset \mathfrak{gl}(\Omega)$ by the representation of $\mathcal{G}_0$ in $Gl(\Omega)$. If $X \in j\mathfrak{l}$, then $\mathrm{ad}\,X - (\mathrm{ad}\,X)' \in \mathfrak{k}$, and by varying $X \in j\mathfrak{l}$ we try to find ξ such that $(\mathrm{ad}\,X)\xi = (\mathrm{ad}\,X)'\xi$. In other words we have

(1) $$<[X,\xi],L> = <\xi,[X,L]>\ \forall X \in j\mathfrak{l}\ ,\ \forall L \in \mathfrak{l}\ .$$

We have $j\mathfrak{l} = \sum_1^p j\mathfrak{k}_{\alpha_k} + \sum_{k<m} \mathfrak{k}_{\frac{1}{2}(\alpha_k-\alpha_m)}$, and it is easy to see that $(\mathrm{ad}\ jE_k)' = \mathrm{ad}\ jE_k$, so we try $X \in \mathfrak{k}_{\frac{1}{2}(\alpha_a-\alpha_b)}$. Let $\xi = \Sigma s_k E_k + \sum_{k<m} Y_{km}$, with $s_k \in \mathbb{R}$ and $Y_{km} \in \mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_m)}$. We have then

$$[X,\xi] = s_b[X,E_b] + [X,Y_{ab}] + \sum_{m>b}[X,Y_{bm}] + \sum_{\substack{k<b\\ k\neq a}}[X,Y_{kb}]\ ;$$

the other terms being zero since the sum of two roots is not always a root. For instance is $\frac{1}{2}(\alpha_a-\alpha_b)+\alpha_l$ no root if $l \neq b$, and similarly is $\frac{1}{2}(\alpha_a-\alpha_b)+\frac{1}{2}(\alpha_k+\alpha_m)$ no root if $b \neq k,m$.

If $L = \Sigma t_l E_l + \sum_{u<v} L_{uv}$ is the decomposition of L, then by the orthogonality of the root spaces we have

$$\langle[X,\xi],L\rangle = t_a\langle[X,Y_{ab}],E_a\rangle + s_b\langle[X,E_b],L_{ab}\rangle$$
$$+ \sum_{m>b}\langle[X,Y_{bm}],L_{am}\rangle + \sum_{\substack{k<b\\ k\neq a}}\langle[X,Y_{kb}],L_{ak}\rangle$$

Similarly, $\langle[X,L],\xi\rangle = s_a\langle[X,L_{ab}],E_a\rangle + t_b\langle[X,E_b],Y_{ab}\rangle$
$$+ \sum_{m>b}\langle[X,L_{bm}],Y_{am}\rangle + \sum_{\substack{k<b\\ k\neq a}}\langle[X,L_{kb}],Y_{ak}\rangle\ .$$

Putting $t_a \neq 0$, other $t_c = 0$, $\forall L_{uv} = 0$, we get by (1): $\langle[X,Y_{ab}],E_a\rangle = 0$. Now $[X,Y_{ab}] \in \mathfrak{k}_{\alpha_a}$, so $[X,Y_{ab}] = \lambda E_a$, some λ. We see $\lambda = 0$, so $[X,Y_{ab}] = 0 \quad \forall X \in \mathfrak{k}_{\frac{1}{2}(\alpha_a-\alpha_b)}$. In particular, since $jY_{ab} \in \mathfrak{k}_{\frac{1}{2}(\alpha_a-\alpha_b)}$, we see by (III) in the definition of a j-algebra that $Y_{ab} = 0$. Then (1) reduces to

$$s_b\langle[X,E_b],L_{ab}\rangle = s_a\langle[X,L_{ab}],E_a\rangle\ ,$$
$$\forall L_{ab} \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)},\ \forall X \in \mathfrak{k}_{\frac{1}{2}(\alpha_a-\alpha_b)}\ , \quad a<b\ .$$

Now by axiom (II) for j-algebras, we have $0 = [X,E_b] + j[jX,E_b] + j[X,jE_b] - [jX,jE_b] = [X,E_b] + 0 - j(-\frac{1}{2}X) + \frac{1}{2}jX = [X,E_b] + jX$, since $\frac{1}{2}(\alpha_a+\alpha_b)+\alpha_b$ is no root. So

(2) $$j[X,E_b] = X \quad \text{for} \quad X \in \mathfrak{h}_{\frac{1}{2}(\alpha_a-\alpha_b)} .$$

Then $\langle [X,E_b],L_{ab}\rangle = \omega[j[X,E_b],L_{ab}] = \omega[X,L_{ab}]$, and similarly $\langle [X,L_{ab}],E_a\rangle = \omega[jE_a,[X,L_{ab}]] = \omega[X,L_{ab}]$, since $[X,L_{ab}] = \lambda E_a$, some λ , and hence $[jE_a,[X,L_{ab}]] = \lambda[jE_a,E_a] = \lambda E_a = [X,L_{ab}]$. So (1) is $(s_b-s_a)\omega[X,L_{ab}] = 0 \ \forall X \in \mathfrak{h}_{\frac{1}{2}(\alpha_a-\alpha_b)}$, $\forall L_{ab} \in \mathfrak{h}_{\frac{1}{2}(\alpha_a+\alpha_b)}$, $a<b$. In particular, $X = jL_{ab}$ gives $(s_b-s_a)|L_{ab}|^2 = 0$ $\forall L_{ab} \in \mathfrak{h}_{\frac{1}{2}(\alpha_a+\alpha_b)}$. So

(3) $$\mathfrak{h}_{\frac{1}{2}(\alpha_a+\alpha_b)} \neq (0) \Rightarrow s_a = s_b .$$

Suppose now that $\mathcal{D}$ is indecomposable, i.e. not biholomorphic to a product of two bounded, homogeneous domains. Then ([4],[5]) also Ω is indecomposable, i.e. not a product of two cones. If not $s_a = s_b \ \forall a,b$, then by (3) the integers $1,\dots,p$ are divided into several groups $J_1,\dots,J_r$, say, such that $k \in J_{i_1}$, $m \in J_{i_2}$, $i_1 \neq i_2$, implies $\mathfrak{h}_{\frac{1}{2}(\alpha_k+\alpha_m)} = (0)$. Then $\mathfrak{l} = \mathfrak{l}_1 \oplus\cdots\oplus \mathfrak{l}_r$, where $\mathfrak{l}_\nu = \sum_{k\in J_\nu} \mathfrak{h}_{\alpha_k} + \sum_{\substack{k<m \\ k,m\in J_\nu}} \mathfrak{h}_{\frac{1}{2}(\alpha_k+\alpha_m)}$, and similarly then $j\mathfrak{l} = j\mathfrak{l}_1 \oplus\cdots\oplus j\mathfrak{l}_r$, Lie algebra direct sum. (Use the fact that a sum of two roots is not always a root.) Further, writing $E = E^{(1)}+\cdots+E^{(r)}$ with $E^{(\nu)} = \sum_{k\in J_\nu} E_k$, we have $(\operatorname{ad} j\mathfrak{l}_\mu)(E^{(\nu)}) = 0$ for $\mu \neq \nu$. We see that $\Omega = \Omega_1 \times\cdots\times \Omega_r$, where $\Omega_\nu = \{gE^{(\nu)} | g \in \mathcal{G}_0^{(\nu)}\}$, $\mathcal{G}_0^{(\nu)}$ being the simply connected group corresponding to $j\mathfrak{l}_\nu$. (See above construction of Ω .) We have a contradiction to the indecomposability of $\mathcal{D}$, hence $s_1 = \cdots = s_p =: s$, i.e. $\xi = sE$. Since Ω is a cone with vertex at the origin, the point E will do as a base point, just as good as ξ . So we take E as a base point.

§2. Two important mappings: R and T.

We now compute the mapping $T:\mathfrak{l} \to \mathfrak{p}$, defined by $T_a E = a$. Observe that $\mathfrak{p} \subset \mathfrak{gl}(\Omega) \subset \mathfrak{gl}(\mathfrak{l})$, and that $\mathfrak{p} \ni X \to XE \in \mathfrak{l}$ is a linear isomorphism. Indeed, since Ω is homogeneous, we have $\dim \mathfrak{p} = \dim \Omega = \dim \mathfrak{l}$, and also $XE = 0$ for $X \in \mathfrak{gl}(\Omega)$ implies $X \in \mathfrak{k}$, since $(\exp tX)E = \sum_{i=0}^{\infty} \frac{t^i X^i E}{i!} = E$ $\forall t \in \mathbb{R}$ in this case. So T is well-defined. (See also [10],[11].)

We stated above that $(\mathrm{ad}\ jE_k)' = \mathrm{ad}\ jE_k$. Indeed, we have:

(i) $<[jE_k,E_l],E_m> = \delta_{kl}<E_l,E_m> = \delta_{kl}\delta_{lm}$, and

$<E_l,[jE_k,E_m]> = \delta_{km}<E_l,E_m> = \delta_{km}\delta_{lm} = \delta_{kl}\delta_{lm}$,

where δ_{kl} is the Kronecker delta.

(ii) If $Z \in \mathfrak{k}_{\frac{1}{2}(\alpha_s+\alpha_m)}$, $s<m$, then

$<[jE_k,E_l],Z> = \delta_{kl}<E_l,Z> = 0$

$<E_l,[jE_k,Z]> = \frac{1}{2}(\alpha_s+\alpha_m)(jE_k)<E_l,Z> = 0$,

(iii) If also $Y \in \mathfrak{k}_{\frac{1}{2}(\alpha_t+\alpha_n)}$, $t<n$, then

$<[jE_k,Y],Z> = \frac{1}{2}(\alpha_t+\alpha_n)(jE_k)<Y,Z>$ $(= 0$ if $(t,n) \neq (s,m))$,

$<Y,[jE_k,Z]> = \frac{1}{2}(\alpha_t+\alpha_n)(jE_k)<Y,Z>$ $(= 0$ if $(t,n) \neq (s,m))$.

So $\mathrm{ad}_{\mathfrak{l}}\ jE_k \in \mathfrak{p}$. Further $[jE_k,E] = E_k$. Hence

(4) $$T_{E_k} = \mathrm{ad}\ jE_k : \mathfrak{l} \to \mathfrak{l}.$$

Now let $X := \mathrm{ad}\, jL_{ab} : \mathfrak{l} \to \mathfrak{l}$ with $L_{ab} \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)}$, $a<b$.
We want to calculate $\frac{1}{2}(X+X')E$.
One computes that for $s_k \in \mathbb{R}$ and $Y_{km} \in \mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_m)}$, we have

(5) $$\begin{aligned}&<(X+X')E\ ,\ \Sigma s_k E_k + \sum_{k<m} Y_{km}> \\ &= <[jL_{ab},E_b],Y_{ab}> + <E_a,[jL_{ab},Y_{ab}]>,\end{aligned}$$

by using the description of roots and the orthogonality of root spaces.

Just as we proved (2), or by (2), we can prove

$$(6) \qquad [jL_{ab},E_b] = L_{ab} \quad \text{for} \quad L_{ab} \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)} .$$

Further $[jE_a,[jL_{ab},Y_{ab}]] = [jL_{ab},Y_{ab}]$, since the last bracket lies in $\mathfrak{h}_{\alpha_a}$. So $\langle E_a,[jL_{ab},Y_{ab}]\rangle = \omega[jE_a,[jL_{ab},Y_{ab}]] = \omega[jL_{ab},Y_{ab}] = \langle L_{ab},Y_{ab}\rangle$. Putting this and (6) into (5), we get $\langle (X+X')E \, , \, \Sigma s_k E_k + \sum_{k<m} Y_{ab}\rangle = 2\langle L_{ab},Y_{ab}\rangle$. So $\frac{1}{2}(X+X')E = L_{ab}$, by the orthogonality of root spaces.

This means, since $X+X' \in \mathfrak{p}$, that

$$(7) \qquad T_{L_{ab}} = \tfrac{1}{2}[\mathrm{ad}\, jL_{ab} + (\mathrm{ad}\, jL_{ab})'] : \mathfrak{l} \to \mathfrak{l} \, ,$$

where transpose ' is with respect to $\langle , \rangle$ on $\mathfrak{l}$.

Now (4) can also be written in this form, so generally

$$(8) \qquad T_L = \tfrac{1}{2}[\mathrm{ad}\, jL + (\mathrm{ad}\, jL)'] : \mathfrak{l} \to \mathfrak{l} \quad \text{for} \quad L \in \mathfrak{l} .$$

Since the computation of Satake's map $R: \mathfrak{l} \to \mathrm{End}_{\mathbb{C}}(\mathfrak{u})$ is similar, and since we need it later, we compute it now. The map is defined by $2\langle E,F(R_L u,v)\rangle = \langle L,F(u,v)\rangle$, where F is the above defining map for $\mathcal{D}$. Here $\langle , \rangle$ on $\mathfrak{l}_{\mathbb{C}}$ is the bilinear extension of $\langle , \rangle$ on $\mathfrak{l}$. Since R_L is $\mathbb{C}$-linear, we have $R_L ju = jR_L u$ for $u \in \mathfrak{u}$. And then

$$4\langle L,F(u,v)\rangle = \langle L,[ju,v]\rangle + i\langle L,[u,v]\rangle ,$$

$$8\langle E,F(R_L u,v)\rangle = 2\langle E,[R_L ju,v]\rangle + 2i\langle E,[R_L u,v]\rangle .$$

Replacing u by ju , we see that

$$(9) \qquad 2\langle E,[R_L u,v]\rangle = \langle L,[u,v]\rangle \;\forall u,v \in \mathfrak{u}$$

characterizes R_L .

Let $L = \sum_l s_l E_l + \sum_{k<m} L_{km}$ be the decomposition of $L \in \mathfrak{l}$, and let $u = \Sigma u_k$, $v = \Sigma v_k$, $w := R_L u = \Sigma w_k \in \mathfrak{u}$ with $u_k,v_k,w_k \in \mathfrak{k}_{\frac{1}{2}\alpha_k}$.

Then (9) is

(10) $$2\Sigma\langle E_l,[w_l,v_l]\rangle = \Sigma t_l\langle E_l,[u_l,v_l]\rangle + \sum_{k<m}\langle L_{km},[u_k,v_m]+[u_m,v_k]\rangle$$

Here we put first $L = E_k$, and get

$$2\Sigma\langle E_l,[w_l,v_l]\rangle = \langle E_k,[u_k,v_k]\rangle \ \forall v \ .$$

If $m \neq k$ and $v_l = 0$ for $l \neq m$, then $2\langle E_m,[w_m,v_m]\rangle = 0 \ \forall v_m$. Since $[w_m,v_m]$ is proportional to E_m , we see $[w_m,v_m] = 0 \ \forall v_m$. Then $v_m = jw_m$ implies $0 = \omega(0) = \omega([w_m,jw_m])$, so $w_m = 0$, (axiom (III)). So $2\langle E_k,[w_k,v_k]\rangle = \langle E_k,[u_k,v_k]\rangle \ \forall v_k$, which means, as above, that $[w_k-\frac{1}{2}u_k,v_k] = 0 \ \forall v_k$, and hence that $w_k = \frac{1}{2}u_k$. So

(11) $$R_{E_k}(\Sigma u_l) = \tfrac{1}{2}u_k \ .$$

Now let $L = L_{km}$, $k < m$. Then $2\Sigma\langle E_l,[w_l,v_l]\rangle =$ $\langle L_{km},[u_k,v_m]+[u_m,v_k]\rangle \ \forall v$. Let $s \neq k,m$ and $v_l = 0$ for $l \neq s$. Then $2\langle E_s,[w_s,v_s]\rangle = 0 \ \forall v_s$. As above, we conclude that $w_s = 0$. So $2\langle E_k,[w_k,v_k]\rangle + 2\langle E_m,[w_m,v_m]\rangle = \langle L_{km},[u_k,v_m]+[u_m,v_k]\rangle$. Then $v_m = 0$ implies $2\langle E_k,[w_k,v_k]\rangle = \langle L_{km},[u_m,v_k]\rangle$. Let $[w_k,v_k] = \lambda E_k$. Then $\langle E_k,[w_k,v_k]\rangle = \omega[jE_k,\lambda E_k] = \lambda\omega(E_k) =$ $\omega([w_k,v_k]) = -\omega([j^2w_k,v_k]) = -\langle jw_k,v_k\rangle$. Further $[jL_{k,m},[u_m,v_k]] = [[jL_{km},u_m],v_k]$ since $\frac{1}{2}(\alpha_k-\alpha_m)+\frac{1}{2}\alpha_k$ is no root. And, by using axiom (II) and the form of the roots, we get

(12) $$[jL_{km},u_m] = -j[jL_{km},ju_m] \quad \text{for } L_{km}\in \mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_m)},\ u_m\in\mathfrak{k}_{\frac{1}{2}\alpha_m} \ .$$

Putting all this into the equation for w_k , we get

$$-2\langle jw_k,v_k\rangle = \omega[jL_{km},[u_m,v_k]] = -\langle [jL_{km},ju_m],v_k\rangle \ \forall v_k \ .$$

So $jw_k = \frac{1}{2}[jL_{km},ju_m] = \frac{1}{2}j[jL_{km},u_m]$, (see (12)). Hence

(13) $$w_k = \tfrac{1}{2}[jL_{km},u_m] \ .$$

If, instead, we put $v_k = 0$, then we get an equation for w_m, and a similar calculation gives

(14) $\quad w_m = \frac{1}{2}(\operatorname{ad} jL_{km})'(u_k)$, where $'$ is transpose with respect to $<,>$ on $\mathcal{U}$.

By the form of the roots, we see that (13) can be written

$w_k = \frac{1}{2}[jL_{km}, u]$, and likewise (take the inner product with an arbitrary element of $\mathcal{U}$) $w_m = \frac{1}{2}(\operatorname{ad} jL_{km})'(u)$. So

(15) $$R_{L_{km}} = \tfrac{1}{2}[\operatorname{ad} jL_{km} + (\operatorname{ad} jL_{km})'] .$$

Also (11) can be written in this form. For $[jE_k, u] = \frac{1}{2}u_k$, and similarly $(\operatorname{ad} jE_k)'(u) = \frac{1}{2}u_k$ since $<(\operatorname{ad} jE_k)'(u), v> = <u, [jE_k, v]>$ $= \frac{1}{2}<u, v_k> = <\frac{1}{2}u_k, v> \;\forall v \in \mathcal{U}$. Hence

(16) $$R_L = \tfrac{1}{2}[\operatorname{ad} jL + (\operatorname{ad} jL)'] : \mathcal{U} \to \mathcal{U} \quad \forall L \in \mathcal{L},$$

where $'$ is the transpose with respect to $<,>$ on $\mathcal{U}$.
(Compare with (8).)

§3. Jordan Structure

If the homogeneous cone Ω is self-adjoint with respect to $<,>$, then $\mathcal{L}$ is a Jordan algebra ([10],[11]) with the product $ab := T_a b$. We have then, by (4), $E_k E_l = T_{E_k} E_l = [jE_k, E_l] = \delta_{kl} E_l$, where δ_{kl} is the Kronecker delta, as before.

Hence the E_k's are orthogonal idempotents. Similarly $L_{ab}E_k$

move up to (or new line)

$= E_k L_{ab} = T_{E_k} L_{ab} = [jE_k, L_{ab}] = \begin{cases} 0 & \text{if } k \neq a,b \\ \frac{1}{2}L_{ab} & \text{if } k = a,b \end{cases}$, where $L_{ab} \in \mathcal{L}_{\frac{1}{2}(\alpha_a+\alpha_b)}$.

And $L_{ab}\tilde{L}_{km} = \frac{1}{2}[\operatorname{ad} jL_{ab} + (\operatorname{ad} jL_{ab})']\tilde{L}_{km}$ for $\tilde{L}_{km} \in \mathcal{L}_{\frac{1}{2}(\alpha_k+\alpha_m)}$.

Now $[jL_{ab}, \tilde{L}_{km}] = 0$ unless $b = k,m$, and similarly $(\operatorname{ad} jL_{ab})'\tilde{L}_{km} = 0$ unless $a = k,m$, as one sees by taking inner products with arbitrary elements of $\mathcal{L}$. We then get:

$$(17)\quad \begin{aligned}
&\text{(i)} && L_{ab}\widetilde{L}_{ma} = \tfrac{1}{2}(\operatorname{ad} jL_{ab})'\widetilde{L}_{ma} \in \mathfrak{k}_{\frac{1}{2}(\alpha_m+\alpha_b)} \\
&\text{(ii)} && L_{ab}\widetilde{L}_{am} = \tfrac{1}{2}(\operatorname{ad} jL_{ab})'\widetilde{L}_{am} \in \mathfrak{k}_{\frac{1}{2}(\alpha_m+\alpha_b)} && \text{for } b \neq m \\
&\text{(iii)} && L_{ab}\widetilde{L}_{mb} = \tfrac{1}{2}[jL_{ab},\widetilde{L}_{mb}] \in \mathfrak{k}_{\frac{1}{2}(\alpha_m+\alpha_a)} && \text{for } a \neq m \\
&\text{(iv)} && L_{ab}\widetilde{L}_{bm} = \tfrac{1}{2}[jL_{ab},\widetilde{L}_{bm}] \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_m)} \\
&\text{(v)} && L_{ab}\widetilde{L}_{ab} = \tfrac{1}{2}[[jL_{ab},\widetilde{L}_{ab}]+(\operatorname{ad} jL_{ab})'\widetilde{L}_{ab}] \in \mathfrak{k}_{\alpha_a} + \mathfrak{k}_{\alpha_b} \\
&\text{(vi)} && L_{ab}\widetilde{L}_{km} = 0 \quad \text{otherwise .}
\end{aligned}$$

In (v) we let $[jL_{ab},\widetilde{L}_{ab}] = \lambda E_a$ and $(\operatorname{ad} jL_{ab})'\widetilde{L}_{ab} = \mu E_b$. Then $\langle L_{ab},\widetilde{L}_{ab}\rangle = \omega[jL_{ab},\widetilde{L}_{ab}] = \lambda\omega(E_a)$ and $\mu\omega(E_b) = \mu\omega[jE_b,E_b] = \mu\langle E_b,E_b\rangle = \langle E_b,(\operatorname{ad} jL_{ab})'\widetilde{L}_{ab}\rangle = \langle [jL_{ab},E_b],\widetilde{L}_{ab}\rangle = \langle L_{ab},\widetilde{L}_{ab}\rangle$ by (6). Hence

$$L_{ab}\widetilde{L}_{ab} = \tfrac{1}{2}\langle L_{ab},\widetilde{L}_{ab}\rangle[\omega(E_a)^{-1}E_a+\omega(E_b)^{-1}E_b] \ . \tag{18}$$

In fact we have, for $Y_{km} \in \mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_m)}$, $s_l \in \mathbb{R}$:

$$(19)\quad \begin{aligned}
&(\operatorname{ad} jL_{ab})'(\sum_l s_lE_l + \sum_{k<m} Y_{km}) = (\operatorname{ad} jL_{ab})'Y_{ab} + s_a(\operatorname{ad} jL_{ab})'E_a + \sum_{\substack{m\neq b\\ m>a}}(\operatorname{ad} jL_{ab})'Y_{am} \\
&+ \sum_{m<a}(\operatorname{ad} jL_{ab})'Y_{ma} \ , \quad \text{with } 1^{\text{st}} \text{ term in } \mathfrak{k}_{\alpha_b} ,\ 2^{\text{nd}} \text{ term in } \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)} , \\
&\qquad\qquad \text{and the other terms as in (17) above.}
\end{aligned}$$

The Jordan identity $L_{ab}^2(L_{ab}E_b) = L_{ab}(L_{ab}^2E_b)$ gives us a relation for the $\omega(E_k)$'s . Indeed, $L_{ab}^2(L_{ab}E_b) = \frac{1}{4}|L_{ab}|^2[\omega(E_a)^{-1}E_a+\omega(E_b)^{-1}E_b]L_{ab}$ $= \frac{1}{8}|L_{ab}|^2[\omega(E_a)^{-1}+\omega(E_b)^{-1}]L_{ab}$, and $L_{ab}(L_{ab}^2E_b)$ $= \frac{1}{2}|L_{ab}|^2L_{ab}[\omega(E_a)^{-1}E_aE_b + \omega(E_b)^{-1}E_b^2] = \frac{1}{2}|L_{ab}|^2\omega(E_b)^{-1}L_{ab}E_b$ $= \frac{1}{4}|L_{ab}|^2\omega(E_b)^{-1}L_{ab}$. So if $L_{ab} \neq 0$, then $\frac{1}{2}[\omega(E_a)^{-1}+\omega(E_b)^{-1}]$ $= \omega(E_b)^{-1}$, i.e.

$$\mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)} \neq (0) \quad \text{implies} \quad \omega(E_a) = \omega(E_b) \ . \tag{20}$$

If $\mathcal{D}$ is indecomposable, then the argument that gave us the base point of Ω also gives us:

$$\text{(21)}\qquad \begin{gathered}\mathcal{D} \text{ indecomposable and } \Omega \text{ self-adjoint with respect to } \langle,\rangle \\ \text{implies } \omega(E_1) = \cdots = \omega(E_p) =: \kappa > 0 \,, \\ \text{and } L_{ab}\tilde{L}_{ab} = \tfrac{1}{2}\kappa^{-1}\langle L_{ab},\tilde{L}_{ab}\rangle(E_a+E_b) \,.\end{gathered}$$

The commutativity of the Jordan product gives us the contitions:

$$\text{(22)}\qquad \begin{array}{lll} \text{(i)} & (\operatorname{ad} jL_{ab})'\,\tilde{L}_{ma} = [j\tilde{L}_{ma},L_{ab}] & \\ \text{(ii)} & (\operatorname{ad} jL_{ab})'\,\tilde{L}_{am} = (\operatorname{ad} j\tilde{L}_{am})'\,L_{ab}\,, & b \neq m \\ \text{(iii)} & [jL_{ab},\tilde{L}_{mb}] = [j\tilde{L}_{mb},L_{ab}]\,, & a \neq m \end{array}$$

These conditions are, however, always satisfied for a normal j-algebra; the last one by axiom (II), and the first two are easily verified by taking inner products. (One shows for instance (i), and then that (i) implies (ii).)

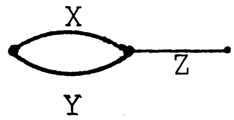
To formulate a proposition, we need some terminology: Let us represent an element $X \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)}$ by a line $\overset{a}{\bullet}\!\underset{X}{\text{——}}\!\overset{b}{\bullet}$ with indices at the vertices to indicate the root space X is taken from. When forming a product **XY** (see below), we write $\overset{a}{\bullet}\!\underset{X}{\text{——}}\!\overset{b}{\bullet}\!\underset{Y}{\text{——}}\!\overset{c}{\bullet}$ if $Y \in \mathfrak{k}_{\frac{1}{2}(\alpha_b+\alpha_c)}$ and $a \neq c$, etc.. We can also have a product $a \underset{Y}{\overset{X}{=}} b$ if X and Y lie in the same root space. The first product is to be an element of $\mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_c)}$, the last an element of $\mathfrak{k}_{\alpha_a} + \mathfrak{k}_{\alpha_b}$. We also drop the indices at the vertices of the graphs if we are interested only in how the elements are "connected". We then have

<u>Proposition 1</u>

Let $(\mathfrak{g} = \mathfrak{t} + j\mathfrak{t} + \mathfrak{u}, j, \omega)$ be a normal j-algebra. Define a product on $\mathcal{L}$ by $E_k E_l := \delta_{kl} E_l$, $E_k L_{ab} = L_{ab} E_k := \begin{cases} 0 & \text{if } k \neq a,b \\ \frac{1}{2} L_{ab} & \text{if } k = a,b \end{cases}$, $(L_{ab} \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)})$, the equation in (21) with some $\kappa > 0$, and the equations (17 i, ii, iii, iv, vi). Extend this product by linearity to $\mathcal{L}$. Then $\mathcal{L}$ is a compact (formally real) Jordan algebra with this product if and only if the conditions (A) and (B) below are satisfied. If so, the unit element is $E = E_1 + \cdots + E_p$, the E_k's being orthogonal idempotents.

Conditions:

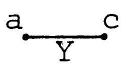
(A) For elements connected as  , we have $(XY)Z = X(YZ)$.
(A partial associativity condition)

(B) For elements connected as  , we have

$$X(YZ) + Y(XZ) = (XY)Z \text{ , i.e.}$$

$$X(YZ) + Y(XZ) = \tfrac{1}{4}\kappa^{-1}\langle X,Y\rangle Z \text{ .}$$

<u>Remarks</u>: (1) Condition (A) is always satisfied in certain cases: Let $X \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)}$, $Z \in \mathfrak{k}_{\frac{1}{2}(\alpha_c+\alpha_d)}$. If the lines below represent Y , then (A) is satisfied in all but the 4 heavily drawn cases:


$a < b < c < d$, 
$a < c < b < d$, 
$a < c < d < b$. As an example, consider the case  in the first figure. We have $(XY)Z = \frac{1}{4}(\mathrm{ad}\, jZ)'(\mathrm{ad}\, jX)'Y$, using (22) and the statement following it. And similarly $X(YZ) = \frac{1}{4}(\mathrm{ad}\, jX)'(\mathrm{ad}\, jZ)'Y$. These elements lie in $\mathfrak{k}_{\frac{1}{2}(\alpha_b+\alpha_d)}$, so let $W \in \mathfrak{k}_{\frac{1}{2}(\alpha_b+\alpha_d)}$ too. Then we have to check that $\langle [jX,[jZ,W]],Y\rangle = \langle [jZ,[jX,W]],Y\rangle$. We have $[jX,[jZ,W]] = [jZ,[jX,W]]$, using the form of the roots, proving the statement. The other cases are similar.

(2) If $X,Y \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)}$, $a<b$, and $Z \in \mathfrak{k}_{\frac{1}{2}(\alpha_c+\alpha_d)}$, $c<d$, then condition (B) is satisfied if $d=a$ or $c=b$. This is also easy to prove. Then there are 4 cases left also for (B).

<u>Indication of proof of Proposition 1</u>: Unfortunately, the complete proof seems to require the application of brutal force to check numerous cases (see below), at least the way the author proved it. But we indicate what has to be done. First of all, the product defined is commutative, as is seen by (22) and the statement following it. The best way then seems to be to use Vinberg's lemma ([11]): Let $A_X: Y \to XY$ and $\sigma(X) := \operatorname{tr} A_X$. If the bilinear form $(X,Y) := \sigma(XY) = \operatorname{tr} A_{XY}$ is positive definite on $\mathfrak{l}$, and if

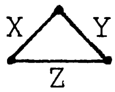
$$[[A_X,A_Y],A_Z] = A_{[X,Z,Y]} \quad \forall X,Y,Z \in \mathfrak{l}, \tag{23}$$

where $[X,Z,Y] = X(ZY)-(XZ)Y$, then $\mathfrak{l}$ is a compact (hence formally real) Jordan algebra. (Compact means exactly that $(\,,\,)$ is positive definite.) The converse is also true: (23) is satisfied by a Jordan algebra. So we first prove that $(\,,\,)>0$, then we check (23) and get the necessary and sufficient conditions (A) and (B): Let $X \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)}$, $a<b$. For $Y \in \mathfrak{k}_\alpha \subset \mathfrak{l}$ one sees, by the multiplication table, that XY has no component in $\mathfrak{k}_\alpha$. Hence $\sigma(X)=0$. Further, the multiplication properties of E_k shows that $\sigma(E_k) = 1 + \frac{1}{2}\sum_{l<k} \dim \mathfrak{k}_{\frac{1}{2}(\alpha_l+\alpha_k)} + \frac{1}{2}\sum_{l>k} \dim \mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_l)} > 0$. ($1=\dim \mathfrak{k}_{\alpha_k}$). Now for $s_k \in \mathbb{R}$, $L_{km} \in \mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_m)}$, we have $(\sum_k s_k E_k + \sum_{k<m} L_{km})^2 = \sum_k s_k^2 E_k + \sum_{k<m}(s_k+s_m)L_{km} + \sum_{k<m} \frac{1}{2}\kappa^{-1}|L_{km}|^2(E_k+E_m) + X$, where $X \in \sum_{k<m} \mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_m)}$. Then $(\sum_k s_k E_k + \sum_{k<m} L_{km}, \sum_k s_k E_k + \sum_{k<m} L_{km}) = \sum_{k=1}^{p} \{s_k^2 + \frac{1}{2}\kappa^{-1}\sum_{l<k}|L_{lk}|^2 + \frac{1}{2}\kappa^{-1}\sum_{l>k}|L_{kl}|^2\}\sigma(E_k) > 0$ if $\sum_k s_k E_k + \sum_{k<m} L_{km} \neq 0$. So $(\,,\,)>0$. The identity (23) means that for all $X,Y,Z,W \in \mathfrak{l}$, we have

(24) $\quad X(Y(ZW))-Y(X(ZW)) = Z[X(YW)-Y(XW)]+W[X(YZ)-Y(XZ)]$,

using the commutativity of the product.
We check (24) in different cases: An element of $\mathfrak{k}_{\alpha_k}$ is said to be of "type I", and an element of $\mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_m)}$, $k<m$, is said to be of "type II". The identity (24) is anti-symmetric in X and Y , and symmetric in Z and W , and therefore it suffices to check the cases indicated in the following table:

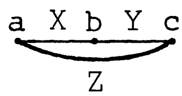
Case	1	2	3	4	5	6	7	8	9
X	I	I	I	I	I	I	II	II	II
Y	I	I	I	II	II	II	II	II	II
Z	I	I	II	I	I	II	I	I	II
W	I	II	II	I	II	II	I	II	II

This is straightforward but spacetime consuming, so we just indicate what happens. Only the cases 6, 8 and 9, especially the last, are somewhat complicated. In cases 6 and 8 the identity (24) is satisfied because of conditions (A) and (B), and because we have the following:
For elements X,Y,Z connected as 

, we have

(25) $\qquad \langle XY,Z\rangle = \langle YZ,X\rangle = \langle ZX,Y\rangle$.

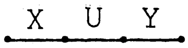
a<b<c.

Proof: Let the elements be as indicated in 

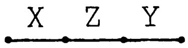
, Then by axiom (II) and the form of roots, we have $j[jX,Y] = [jX,jY]$. Therefore $\langle XY,Z\rangle = \frac{1}{2}\omega[j[jX,Y],Z] = \frac{1}{2}\omega[[jX,jY],Z] = \frac{1}{2}\omega[jX,[jY,Z]]$ $= \langle X,YZ\rangle$, since $[jX,Z] = 0$ because of the form of the roots. Similarly, $\langle XY,Z\rangle = \frac{1}{2}\langle [jX,Y],Z\rangle = \frac{1}{2}\langle Y,(\mathrm{ad}\,jX)'Z\rangle = \langle Y,XZ\rangle$. (The first case could also be proved like this.) q.e.d.

In case 9 we get several conditions, but they are all satisfied because of (25) and conditions (A) and (B):

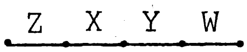
<u>Two-term conditions</u>:

(i) $X(YU)-Y(XU) = 0$ with X,Y and $U:=ZW$ connected as 

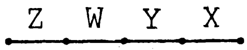
(ii) $Z[X(YW)-Y(XW)] = 0$ with X,Y,W connected as 

(iii) $W[X(YZ)-Y(XZ)] = 0$ with X,Y,Z connected as 

These conditions are all satisfied because of (A).

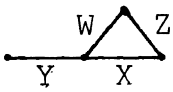
(iv) $Z(X(YW)) = ((ZX)Y)W$ with X,Y,Z,W connected as 

We have $Z(X(YW)) = Z((XY)W) = (Z(XY))W = ((ZX)Y)W$ because of (A).

(v) $X(Y(ZW)) = Z(X(YW))$ with X,Y,Z,W connected as 

Interchange X and W in (iv) and observe that $W(YX) = X(YW)$ by (A).

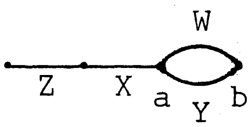
<u>Three-term conditions</u>:

(i) $X(Y(ZW))-Y(X(ZW))+W(Y(XZ)) = 0$ with diagram 

We have, by (A) and (B): $X(Y(ZW))+W(Y(XZ))-Y(X(ZW))$

$$= X((ZW)Y)+W(Z(XY))-Y(X(ZW))$$

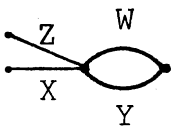
$$= X((ZW)Y)+(ZW)(XY)-Y(X(ZW)) = 0$$

(ii) $Z[Y(XW)-X(YW)]+W(Y(XZ)) = 0$ with diagram  .

We have, by (A) and (B): $Z[Y(XW)-X(YW)]+W(Y(XZ))$

$$= Y(Z(XW))+W(Y(XZ))-Z(\tfrac{1}{2}\kappa^{-1}\langle Y,W\rangle X(E_a+E_b))$$

$$= Y(W(ZX))+W(Y(ZX))-\tfrac{1}{4}\kappa^{-1}\langle Y,W\rangle ZX = 0$$

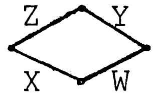
(iii) $Z[Y(XW)-X(YW)]+X(Y(ZW)) = 0$ with diagram 

We have, by (A) and (B): $Z[Y(XW)-X(YW)]+X(Y(ZW))$

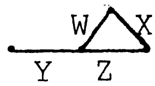
$$= Z[Y(XW)-X(YW)]+(ZW)(YX) = Z[Y(XW)-X(YW)]+Z(W(YX))$$

$$= Z[Y(WX)+W(YX)-X(YW)] = 0$$

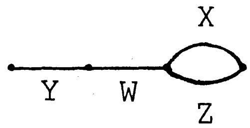
<u>Four-term conditions</u>:

(i) $Z[X(YW)-Y(XW)]+W[X(YZ)-Y(XZ)] = 0$ with diagram 

Here each bracket vanishes, by (A).

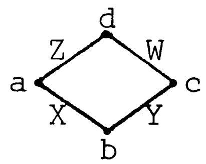
(ii) $Z[X(YW)-Y(XW)]+W[X(YZ)-Y(XZ)] = 0$ with diagram 

Again each bracket vanishes, by (A).

(iii) $X(Y(ZW))-Y(X(ZW)) = Z[X(YW)-Y(XW)]$ with diagram 

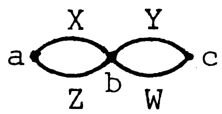
Each side vanishes, by (A).

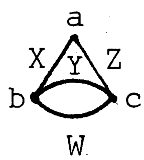
(iv) $X(Y(ZW))-Y(X(ZW))$

$= Z(X(YW))-W(Y(XZ))$ with diagram 

We have $X(Y(ZW))-Y(X(ZW)) = \frac{1}{2}\kappa^{-1}[\langle X,Y(ZW)\rangle(E_a+E_b)-\langle Y,X(ZW)\rangle(E_b+E_c)]$ $= \frac{1}{2}\kappa^{-1}\langle XY,ZW\rangle(E_a-E_c)$, by (25). And, by (A) and (25), we have also $Z(X(YW))-W(Y(XZ)) = \frac{1}{2}\kappa^{-1}[\langle Z,X(YW)\rangle(E_a+E_d)-\langle W,Y(XZ)\rangle(E_c+E_d)]$ $= \frac{1}{2}\kappa^{-1}\langle ZW,XY\rangle(E_a-E_c)$.

<u>Six-term conditions</u>:

The identity (24) for the diagrams 

and 

.

<u>First case</u>: $X(Y(ZW))-Y(X(ZW))-Z[X(YW)-Y(XW)]-W[X(YZ)-Y(XZ)]$

$$= \tfrac{1}{2}\kappa^{-1}\{\langle X,Y(ZW)\rangle(E_a+E_b)-\langle Y,X(ZW)\rangle(E_b+E_c)-Z[\langle Y,W\rangle X(E_b+E_c)]$$
$$+\langle Z,Y(XW)\rangle(E_a+E_b)-\langle W,X(YZ)\rangle(E_b+E_c)+W[\langle X,Z\rangle Y(E_a+E_b)]\}$$
$$= \tfrac{1}{2}\kappa^{-1}\{\langle X,Y(ZW)\rangle(E_a+E_b)-\langle Y,X(ZW)\rangle(E_b+E_c)-\tfrac{1}{4}\kappa^{-1}\langle Y,W\rangle\langle Z,X\rangle(E_a+E_b)$$
$$+\langle Z,Y(XW)\rangle(E_a+E_b)-\langle W,X(YZ)\rangle(E_b+E_c)+\tfrac{1}{4}\kappa^{-1}\langle X,Z\rangle\langle W,Y\rangle(E_b+E_c)\}$$
$$= \tfrac{1}{2}\kappa^{-1}\{[\langle X,Y(ZW)\rangle+\langle Z,Y(XW)\rangle-\tfrac{1}{4}\kappa^{-1}\langle X,Z\rangle\langle Y,W\rangle]E_a$$
$$-[\langle Y,X(ZW)\rangle+\langle W,X(YZ)\rangle-\tfrac{1}{4}\kappa^{-1}\langle X,Z\rangle\langle Y,W\rangle]E_c$$
$$+[\langle X,Y(ZW)\rangle-\langle Y,X(ZW)\rangle+\langle Z,Y(XW)\rangle-\langle W,X(YZ)\rangle]E_b\} .$$

We have $\langle X,Y(ZW)\rangle+\langle Z,Y(XW)\rangle = \langle Y,X(ZW)\rangle+\langle Y,Z(XW)\rangle$
$= \langle Y,\frac{1}{4}\kappa^{-1}\langle X,Z\rangle W\rangle = \frac{1}{4}\kappa^{-1}\langle X,Z\rangle\langle Y,W\rangle$, by (25) and (B).

Similarly the second square bracket above vanishes. Finally, $\langle X,Y(ZW)\rangle = \langle Y,X(ZW)\rangle = \langle XY,ZW\rangle$ and $\langle Z,Y(XW)\rangle = \langle W,X(YZ)\rangle = \langle YZ,XW\rangle$, by (25), so also the last square bracket vanishes.

Second case: $Y(X(ZW))+Z(X(YW))+W(X(YZ))-W(Y(XZ))$

$= \frac{1}{2}\kappa^{-1}\{\langle X,ZW\rangle Y(E_a+E_b)+Z[\langle Y,W\rangle X(E_b+E_c)]+\langle X,YZ\rangle W(E_a+E_b)-\langle Y,XZ\rangle W(E_b+E_c)\}$

$= \frac{1}{2}\kappa^{-1}\{\frac{1}{2}\langle X,ZW\rangle Y+\frac{1}{2}\langle Y,W\rangle ZX+\frac{1}{2}\langle Y,XZ\rangle W-\langle Y,XZ\rangle W\}$

$= \frac{1}{4}\kappa^{-1}\{\langle W,XZ\rangle Y-\langle Y,XZ\rangle W+\langle Y,W\rangle ZX\}$, using (25). We have to check if this equals $X(Y(ZW))+Z(Y(XW))$. Now by (B) we have $X(Y(ZW))$ $= X[\frac{1}{2}\kappa^{-1}\langle Y,W\rangle Z-W(YZ)] = \frac{1}{4}\kappa^{-1}\langle Y,W\rangle XZ-X(W(ZY))$, so we have to check if $Z(Y(XW))-X(W(ZY)) = \frac{1}{4}\kappa^{-1}\{\langle W,XZ\rangle Y-\langle Y,XZ\rangle W\}$. Using (B) and (25), the left hand side equals $\{Z(Y(XW))+(XW)(ZY)\}-\{X(W(ZY))+(ZY)(XW)\}$
$= \frac{1}{4}\kappa^{-1}\{\langle Z,XW\rangle Y-\langle X,ZY\rangle W\} = \frac{1}{4}\kappa^{-1}\{\langle W,XZ\rangle Y-\langle Y,XZ\rangle W\}$.

This completes the outline of the proof of the proposition.

We have, by (21), a condition

(C) $\qquad \omega(E_1) = \cdots = \omega(E_p)$, where $p = \dim\mathfrak{k} = \dim[\mathfrak{g},\mathfrak{g}]^{\perp}$.

§4. Quasi-symmetry.

Now let $\mathcal{D} = \mathcal{D}(\Omega,F)$ be a Siegel domain with self-adjoint (with respect to a given inner product) and homogeneous cone Ω . Choose a basepoint $E \in \Omega$ as explained. Following Satake ([10]), $\mathcal{D}$ is called quasi-symmetric if the mappings T and R (see (8) and (16)) satisfy the condition

(Q) $\qquad T_L F(u,v) = F(R_L u,v)+F(u,R_L v) \quad \forall u,v$

for all $L \in$ space where Ω lies.

In our case this means

$$T_L[ju,v]+iT_L[u,v] = \{[jR_L u,v]+[ju,R_L v]\}+i\{[R_L u,v]+[u,R_L v]\} .$$

Since $jR_L u = R_L ju$, we see that (Q) is equivalent to, in our case:

(Q') $\qquad T_L[u,v] = [R_L u,v]+[u,R_L v]\ \forall u,v,L$.

Now $[jL,[u,v]] = [[jL,u],v]+[u,[jL,v]]$ means, together with (8) and (16), that (Q') is equivalent to

(Q") $\qquad (\mathrm{ad}\,jL)'[u,v] = [(\mathrm{ad}\,jL)'u,v]+[u,(\mathrm{ad}\,jL)'v]\ \forall u,v,L$.

Condition (Q') is satisfied for $L = E_k$, by (4) and (11), since

$$R_{E_k}(\sum_l u_l) = \tfrac{1}{2}u_k = [jE_k,u] .$$

Now let $L_{ab} \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)}$, $a < b$. We use (19) and

(26) $\qquad (\mathrm{ad}\,jL)'(\sum_l u_l) = (\mathrm{ad}\,jL_{ab})'u_a \in \mathfrak{k}_{\frac{1}{2}\alpha_b}$ for $u_l \in \mathfrak{k}_{\frac{1}{2}\alpha_l}$,

which follows by taking inner product and observing the form of the roots. Noting that in the decomposition $[u,v] = \sum_l [u_l,v_l] + \sum_{k<m}\{[u_k,v_m]+[u_m,v_k]\}$ we have $[u_l,v_l] \in \mathfrak{k}_{\alpha_l}$ and $[u_k,v_m]+[u_m,v_k] \in \mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_m)}$, we see that (Q") for L_{ab} means:

(Qi) $\qquad (\mathrm{ad}\,jL_{ab})'\{[u_a,v_b] + [u_b,v_a]\} = [(\mathrm{ad}\,jL_{ab})'u_a,v_b] + [u_b,(\mathrm{ad}\,jL_{ab})'v_a] \in \mathfrak{k}_{\alpha_b}$

(Qii) $\qquad (\mathrm{ad}\,jL_{ab})'[u_a,v_a] = [(\mathrm{ad}\,jL_{ab})'u_a,v_a] + [u_a,(\mathrm{ad}\,jL_{ab})'v_a] \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)}$

(Qiii) $\qquad (\mathrm{ad}\,jL_{ab})'\{[u_a,v_m]+[u_m,v_a]\} = [(\mathrm{ad}\,jL_{ab})'u_a,v_m]+[u_m,(\mathrm{ad}\,jL_{ab})'v_a] \in \mathfrak{k}_{\frac{1}{2}(\alpha_b+\alpha_m)}$,

Putting $v_b = 0$ in (i) and $v_m = 0$ in (iii) shows that (i) and (iii) are equivalent to the condition

($\tilde{A}$) $\qquad (\mathrm{ad}\,jL_{ab})'[u_m,v_a] = [u_m,(\mathrm{ad}\,jL_{ab})'v_a] \in \mathfrak{k}_{\frac{1}{2}(\alpha_m+\alpha_b)}$, $m \neq a$.

<u>Lemma 1</u>: Condition ($\tilde{A}$) is always satisfied for $m \geq b$.

Proof. 1) <u>$m = b$</u> . We have $\langle E_b,(\mathrm{ad}\,jL_{ab})'[u_b,v_a]\rangle =$

$<[jL_{ab},E_b],[u_b,v_a]> = <L_{ab},[u_b,v_a]> = \omega[jL_{ab},[u_b,v_a]] =$ $\omega[[jL_{ab},u_b],v_a]$, where we used (6). Further, letting $[u_b,(\mathrm{ad}\,jL_{ab})'v_a] = \lambda E_b$, we have $<E_b,[u_b,(\mathrm{ad}\,jL_{ab})'v_a]> =$ $\lambda\omega[jE_b,E_b] = \lambda\omega(E_b) = \omega[u_b,(\mathrm{ad}\,jL_{ab})'v_a] = -\omega[j^2u_b,(\mathrm{ad}\,jL_{ab})'v_a] =$ $-<ju_b,(\mathrm{ad}\,jL_{ab})'v_a> = -<[jL_{ab},ju_b],v_a> = -<j[jL_{ab},u_b],v_a> =$ $\omega[[jL_{ab},u_b],v_a]$.

2) <u>$m > b$</u> . For $Y_{bm} \in \mathfrak{k}_{\frac{1}{2}(\alpha_b+\alpha_m)}$ it follows from axiom (II) that

(27) $$j[jL_{ab},Y_{bm}] = [jL_{ab},jY_{bm}] .$$

And by (12) we have $j[jY_{bm},u_m] = [jY_{bm},ju_m]$. Using this and (27), a calculation similar to the above shows that

$<Y_{bm},(\mathrm{ad}\,jL_{ab})'[u_m,v_a]> = \omega[[[jL_{ab},jY_{bm}],u_m],v_a] =$
$= <Y_{bm},[u_m,(\mathrm{ad}\,jL_{ab})'v_a]>$. For instance, to prove the second equality, one needs to know that $[jY_{bm},(\mathrm{ad}\,jL_{ab})'v_a] = 0$, which follows because $\frac{1}{2}(\alpha_b-\alpha_m) + \frac{1}{2}\alpha_b$ is no root. q.e.d.

Now since $(\mathrm{ad}\,jL_{ab})'E_a \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)}$, we have for $Y_{ab} \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)}$, observing that $[jL_{ab},Y_{ab}] \in \mathfrak{k}_{\alpha_a}$, that $<Y_{ab},(\mathrm{ad}\,jL_{ab})'E_a> =$ $<[jL_{ab},Y_{ab}],E_a> = \omega[jE_a,[jL_{ab},Y_{ab}]] = \omega[jL_{ab},Y_{ab}] = <Y_{ab},L_{ab}>$. Hence, in analogy to (6),

(28) $$(\mathrm{ad}\,jL_{ab})'E_a = L_{ab} \quad \text{for} \quad L_{ab} \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)} .$$

In the above condition (Qii) we write $[u_a,v_a] = \lambda E_a$. Then by (28) one sees that

$$(\mathrm{ad}\,jL_{ab})'[u_a,v_a] = \lambda L_{ab} , \text{ and since } \omega[u_a,v_a] = \lambda\omega(E_a) = \lambda\kappa,$$

(see (21)), it follows that (Qii) is equivalent to

(Qiv) $$\kappa^{-1}\omega([u_a,v_a])L_{ab} = [(\mathrm{ad}\,jL_{ab})'u_a,v_a]+[u_a,(\mathrm{ad}\,jL_{ab})'v_a] \;\forall u_a,v_a,L_{ab} .$$

Putting here $u_a = jv_a$, we get a left hand side equal to $\kappa^{-1}|v_a|^2L_{ab}$.

Hence we have the

Corollary 1. If $\mathcal{D}(\Omega,F)$ is indecomposable and quasi-symmetric, then

$$\mathfrak{k}_{\frac{1}{2}\alpha_a} \neq (0) \text{ implies } \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)} = [\mathfrak{k}_{\frac{1}{2}\alpha_a}, \mathfrak{k}_{\frac{1}{2}\alpha_b}] \ \forall b > a .$$

We need

Lemma 2. If $w_1 \in \mathfrak{k}_{\frac{1}{2}\alpha_1}$ and $Y_{ab} \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)}$, then

$$<w_b,(\mathrm{ad}\,jY_{ab})'w_a> = <[jw_b,w_a],Y_{ab}> .$$

Proof: Left hand side equals $<[jY_{ab},v_b],v_a> = \omega[j[jY_{ab},v_b],v_a]$ $= \omega[[jY_{ab},jv_b],v_a] = \omega[jY_{ab},[jv_b,v_a]] = <Y_{ab},[jv_b,v_a]>$, using (12) and because $[jY_{ab},v_a] = 0$ since $\frac{1}{2}(\alpha_a-\alpha_b)+\frac{1}{2}\alpha_a$ is **no root.** q.e.d.

Now (Qiv) is equivalent to

$$(29) \qquad \kappa^{-1}\omega[u_a,v_a]<L_{ab},Y_{ab}> = <[(\mathrm{ad}\,jL_{ab})'u_a,v_a] + [u_a,(\mathrm{ad}\,jL_{ab})'v_a],Y_{ab}> ,$$
$$\forall\, u_a,v_a \in \mathfrak{k}_{\frac{1}{2}\alpha_a} ,\ \forall\, L_{ab},Y_{ab} \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)} .$$

The right hand side is $<[j^2(\mathrm{ad}\,jL_{ab})'v_a,u_a] - [j^2(\mathrm{ad}\,jL_{ab})'u_a,v_a],Y_{ab}>$, so, by Lemma 2, (Qiv) is equivalent to

$$(30) \qquad \kappa^{-1}\omega[u_a,v_a]<L_{ab},Y_{ab}> = <j(\mathrm{ad}\,jL_{ab})'v_a,(\mathrm{ad}\,jY_{ab})'u_a> - <j(\mathrm{ad}\,jL_{ab})'u_a,(\mathrm{ad}\,jY_{ab})'v_a> ,\ \forall \text{ everything}.$$

Putting $u_a = jv_a$, then, since $(\mathrm{ad}\,jL_{ab})'$ and $(\mathrm{ad}\,jY_{ab})'$ commute with j, and $<,>$ is j-invariant, we see that (30) implies

$$(31) \qquad \kappa^{-1}|v_a|^2<L_{ab},Y_{ab}> = 2<(\mathrm{ad}\,jL_{ab})'v_a,(\mathrm{ad}\,jY_{ab})'v_a> \ \forall v_a,L_{ab} .$$

Conversely, polarization shows that (31) implies

$$\kappa^{-1}<u_a,v_a><L_{ab},Y_{ab}> = <(\mathrm{ad}\,jL_{ab})'u_a,(\mathrm{ad}\,jY_{ab})'v_a> + <(\mathrm{ad}\,jL_{ab})'v_a,(\mathrm{ad}\,jY_{ab})'u_a> .$$

Replacing here v_a by jv_a, we get back to (30), which is therefore equivalent to (31).

In (31) we put $Y_{ab} = L_{ab}$, and get the condition

(Qv) $\kappa^{-1}|v_a|^2|L_{ab}|^2 = 2|(\mathrm{adj}L_{ab})'v_a|^2$ for $v_a \in \mathfrak{k}_{\frac{1}{2}\alpha_a}$, $L_{ab} \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)}$, $b > a$,

or, equivalently, $\sqrt{2\kappa}|L_{ab}|^{-1}(\mathrm{adj}L_{ab})' : \mathfrak{k}_{\frac{1}{2}\alpha_a} \to \mathfrak{k}_{\frac{1}{2}\alpha_b}$ is an isometry into if $L_{ab} \neq 0$.

Conversely, if (Qv) holds, then polarization in L_{ab} gets us back to (31), which is therefore equivalent to (Qv). So (Qii) is equivalent to (Qv). Now in [9], p. 61, we see that $\sqrt{2\kappa}|L_{ab}|^{-1}\mathrm{adj}L_{ab} : \mathfrak{k}_{\frac{1}{2}\alpha_b} \to \mathfrak{k}_{\frac{1}{2}\alpha_a}$ is an isometry into in general. So $\dim \mathfrak{k}_{\frac{1}{2}\alpha_b} \leq \dim \mathfrak{k}_{\frac{1}{2}\alpha_a}$ for $a < b$ if $\mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)} \neq (0)$. Under (Qv) we have $\dim \mathfrak{k}_{\frac{1}{2}\alpha_a} \leq \dim \mathfrak{k}_{\frac{1}{2}\alpha_b}$ in that case, hence (Qv) implies $\dim \mathfrak{k}_{\frac{1}{2}\alpha_a} = \dim \mathfrak{k}_{\frac{1}{2}\alpha_b}$ if $\mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)} \neq (0)$. Conversely, if we have this equality, then $\sqrt{2\kappa}|L_{ab}|^{-1}\mathrm{adj}L_{ab} : \mathfrak{k}_{\frac{1}{2}\alpha_b} \to \mathfrak{k}_{\frac{1}{2}\alpha_a}$ is an isometry <u>onto</u> for $L_{ab} \neq 0$, and hence, taking transpose, we get (Qv). So (Qii) is equivalent to

(Qvi) $\dim \mathfrak{k}_{\frac{1}{2}\alpha_a} = \dim \mathfrak{k}_{\frac{1}{2}\alpha_b}$ if $\mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)} \neq (0)$.

If $\mathcal{D}$ is indecomposable, then the usual argument (that gave us the base point E) shows that (Qvi), and hence (Qii), are equivalent to

$(\tilde{D})$ $\dim \mathfrak{k}_{\frac{1}{2}\alpha_k}$ is independent of k.

We have

<u>Lemma 3</u>. If $\mathcal{D}$ is indecomposable and satisfies condition $(\tilde{D})$, then $(\tilde{A})$ implies (A) if $\mathcal{U} \neq (0)$.

Proof: By Remark 1 following Proposition 1 we have to check (A) in four cases. They are all similar, so consider the case $a < \widehat{c < b} < d$, with $X \in \mathfrak{k}_{\frac{1}{2}(\alpha_a+\alpha_b)}$, $Y \in \mathfrak{k}_{\frac{1}{2}(\alpha_c+\alpha_b)}$, $Z \in \mathfrak{k}_{\frac{1}{2}(\alpha_c+\alpha_d)}$. Now $\mathcal{U} \neq (0)$ and $(\tilde{D})$ together with Corollary 1, (which is just a consequence of (Qiv), hence of $(\tilde{D})$), show that we can check with $Y = [u_c,u_b]$, where $u_k \in \mathfrak{k}_{\frac{1}{2}\alpha_k}$. Then, using $(\tilde{A})$:

$4X(YZ) = [jX,(\mathrm{adj}Z)'[u_c,u_b]] = [jX,[(\mathrm{adj}Z)'u_c,u_b]] = [(\mathrm{adj}Z)'u_c,[jX,u_b]]$, and

$4(XY)Z = (\mathrm{adj}Z)'[jX,[u_c,u_b]] = (\mathrm{adj}Z)'[u_c,[jX,u_b]] = [(\mathrm{adj}Z)'u_c,[jX,u_b]]$.

q.e.d.

We now recall the definition of quasi-symmetry. A Siegel domain $\mathcal{D}(\Omega,F)$ is called <u>quasi-symmetric</u> if Ω is homogeneous (i.e. if $G(\Omega)$ is transitive on Ω), if Ω is self-adjoint with respect to some inner product $<,>$ on the space containing Ω, and if the mappings T and R, defined earlier, satisfy the condition (Q), where T and R are defined with respect to a base point $\xi \in \Omega$ with the (above mentioned) property that

$$\{X \in \mathfrak{gl}(\Omega) \mid X\xi = 0\} = \{X \in \mathfrak{gl}(\Omega) \mid X' = -X\},$$

X' being the transpose with respect to $<,>$.

<u>Remarks</u>. (1) If $\lambda > 0$, then $\lambda\xi$ is just as good a base point as ξ, and, by their definitions, T and R are replaced by $\frac{1}{\lambda}T$ and $\frac{1}{\lambda}R$, still satisfying (Q) if T and R do.

(2) It is known ([4],[5]) that $\mathcal{D}(\Omega,F)$ is indecomposable if and only if Ω is indecomposable. The indecomposable, homogeneous, self-adjoint cones are known (see for instance [13]), and using, for instance, their description, it is easy to see that for such a cone we have the situation that the stability subgroup of $Gl(\Omega)$ at a point $x \in \Omega$ has exactly the fixpoint set $\{\lambda x \mid \lambda > 0\}$ in Ω. This remark is used in the next lemma.

<u>Lemma 4</u>. If Ω is an indecomposable, homogeneous cone that is self-adjoint with respect to two inner products $<,>_1$ and $<,>_2$ on the space V containing Ω, then there exists an element $\psi \in Gl(\Omega)$ taking one product to the other.

Proof: Each product $<,>_j$ defines a base point $\xi_j \in \Omega$, and we normalize them so that $<\xi_j,\xi_j>_j = 1$. Since Ω is homogeneous there is an element $\varphi \in Gl(\Omega)$ such that $\varphi(\xi_1) = \xi_2$. Let K_j be the stability subgroup of $Gl(\Omega)$ at ξ_j, and let φ^* be the transpose of φ with respect to both products, i.e. $<\varphi^*u,v>_1 = <u,\varphi v>_2$. Also observe that the transpose K_j' of K_j with respect to $<,>_j$ equals K_j. We have $g \in K_1$ if and only if $\varphi g\varphi^{-1} \in K_2$, and taking transpose, we have $g' \in K_1$ if and only if $\varphi^{*-1}g'\varphi^* \in K_2$, where g' is transpose of g with respect to $<,>_1$. Replacing g' by g, we have $g \in K_1$ if and only if $\varphi^{*-1}g\varphi^* \in K_2$. So $\varphi^*\tilde{g}\varphi^{*-1} \in K_1$ if and only if $\tilde{g} \in K_2$. (Replace g by $\tilde{g} = \varphi^{*-1}g\varphi^*$). Also observe that $\varphi^* \in Gl(\Omega)$, since $x \in \Omega$, $y \in \bar{\Omega}\setminus\{0\}$ implies $<\varphi^*x,y>_1 = <x,\varphi y>_2 > 0$ since $\varphi y \in \bar{\Omega}\setminus\{0\}$ and Ω is self-adjoint with respect to $<,>_2$. So, since Ω is also self-adjoint with respect to $<,>_1$, we see that $\varphi^*x \in \Omega$. Then also $\psi := \varphi^*\varphi \in Gl(\Omega)$. We have $\psi K_1\psi^{-1} = \varphi^*\varphi K_1\varphi^{-1}\varphi^{*-1} = \varphi^*K_2\varphi^{*-1} = K_1$ by the above. So $K_1\psi(\xi_1) = \psi K_1\psi^{-1}\psi(\xi_1) = \psi K_1\xi_1 = \psi(\xi_1)$. By Remark (2) above we see that $\psi(\xi_1) = t\xi_1$ for some $t > 0$, i.e. $t^{-1}\psi \in K_1$. Further $t^{-1}\psi$ is symmetric with respect to $<,>_1$ since $<\varphi^*\varphi u,v>_1 = <\varphi u,\varphi v>_2 = <u,\varphi^*\varphi v>_1$. Also elements of K_1 are orthogonal transformations with respect to $<,>_1$. Hence, writing $t^{-1}\psi$ in canonical form with blocks equal to ± 1, $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ along the diagonal, and zeros elsewhere, the symmetry of $t^{-1}\psi$ implies that $\theta = \pm 1$, and hence $t^{-1}\psi$ is diagonal with eigenvalues ± 1. Now $<u,\psi u>_1 = t^{-1}<u,\varphi^*\varphi u>_1$ $= t^{-1}<\varphi u,\varphi u>_2 > 0$ if $u \neq 0$, so all eigenvalues are equal to 1, i.e. $\varphi^*\varphi = t\,\mathrm{Id}$. Then $t = \langle t\xi_1,\xi_1\rangle_1 = \langle\varphi^*\varphi\xi_1,\xi_1\rangle_1 = \langle\varphi\xi_1,\varphi\xi_1\rangle_2 = \langle\xi_2,\xi_2\rangle_2 = 1$, so $\varphi^*\varphi = \mathrm{Id}$. Then $\langle\varphi u,\varphi v\rangle_2 = \langle\varphi^*\varphi u,v\rangle_1 = \langle u,v\rangle_1$ for $u,v \in V$. q.e.d.

This proof was supplied by Harald Hanche-Olsen. (The lemma is probably well known.)

Now a bounded, homogeneous domain $\mathcal{D}$ is biholomorphic to some homogeneous Siegel domain $\mathcal{D}(\Omega,F)$, and $\mathcal{D}$ is called quasi-symmetric if $\mathcal{D}(\Omega,F)$ is quasi-symmetric. This definition is independent of the choice $\mathcal{D}(\Omega,F)$ of representation of $\mathcal{D}$, because of the

Uniqueness Theorem (Kaup-Matsushima-Ochiai) ([6]). Two Siegel domains $\mathcal{D}(\Omega,F)$ and $\mathcal{D}(\Omega',G')$ with $\Omega \subset V$, $\Omega' \subset V'$, $F:U\times U \to V_{\mathbb{C}}$, $F':U'\times U' \to V'_{\mathbb{C}}$ are biholomorphic if and only if there exist a real linear isomorphism $\varphi:V \to V'$ and a complex linear isomorphism $\psi:U \to U'$ such that $\varphi(\Omega) = \Omega'$ and $\varphi_{\mathbb{C}}F(u,v) = F'(\psi u,\psi v)$.

We need a lemma proved in [13]:

Lemma 5. If $\mathcal{D}(\Omega,F)$ is an indecomposable, quasi-symmetric Siegel domain, then Ω is self-adjoint with respect to the Bergman metric on $\mathcal{D}(\Omega,F)$, (which exists; see also the above uniqueness theorem), restricted to the space V containing Ω , considered as a subspace of the tangent space at $(i\xi,0)$, where ξ is a base point (of usual type) in Ω.

(Actually the lemma was not checked in [13] for the case of the exceptional, symmetric tube domain over the cone of positive definite symmetric 3×3-matrices with Cayley number entries.)

Finally, we need a condition on the spaces $\mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_m)}$:

(D) $\quad d_k := \sum_{l\neq k} \dim \mathfrak{k}_{\frac{1}{2}(\alpha_l+\alpha_k)}$ is independent of k .

Remarks: (1) If p, the number of E_k's, equals 3, then (D) is equivalent to the statement that $\dim \mathfrak{k}_{\frac{1}{2}(\alpha_l+\alpha_k)}$ is independent of l,k, $(l \neq k)$.

(2) Another condition that is always satisfied by the dimensions, is: If $k<l<m$, and $\mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_l)}$ and $\mathfrak{k}_{\frac{1}{2}(\alpha_l+\alpha_m)}$ are both non-zero, then $\dim \mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_m)} \geq \max\{\dim \mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_l)}, \dim \mathfrak{k}_{\frac{1}{2}(\alpha_l+\alpha_m)}\}$. This follows from [9], p. 61, (but is not needed here).

We have

<u>Lemma 6</u>. If Ω is indecomposable, and self-adjoint with respect to $<,>$, then (D) is satisfied.

Proof. By the Vinberg-Koecher theorem ([11]), mentioned earlier, we have that Ω is also self-adjoint with respect to the trace product $(X,Y) := \operatorname{tr} A_{XY}$ gotten from the Jordan structure. (See also [1], p. 323). So we can apply Lemma 4. We have $<E,E> = \omega([jE,E]) = \sum_{k=1}^{p} \omega[jE_k,E_k] = \sum_{k=1}^{p} \omega(E_k) = p\kappa$, (see (21)). Hence $e := E/\sqrt{p\kappa}$ is a unit vector with respect to $<,>$. Similarly $(E,E) = \operatorname{tr} A_E = \dim \mathfrak{C} =: d$, so $\tilde{e} := E/\sqrt{d}$ is a unit vector with respect to $(,)$. By (the proof of) Lemma 4, there is $\varphi \in Gl(\Omega)$ such that $\varphi e = \tilde{e}$ and $(\varphi X, \varphi X) = <X,X>$ for all $X \in \mathfrak{C}$. Letting $\mu := \sqrt{d/p\kappa}$, we have $\mu\varphi \in Gl(\Omega)$ and $\mu\varphi e = e$, and hence $\mu\varphi$ and $\mu^{-1}\varphi^{-1}$ are orthogonal transformations with respect to $<,>$. So $(X,X) = <\varphi^{-1}X, \varphi^{-1}X> = \mu^2 <\mu^{-1}\varphi^{-1}X, \mu^{-1}\varphi^{-1}X> = \mu^2 <X,X>$. Now for $X = \sum_{l=1}^{p} s_l E_l + \sum_{k<m} L_{km}$, we have, by the calculation preceeding (24), that $(X,X) = \sum_{l=1}^{p} s_l^2 \sigma(E_l) + \frac{1}{2}\kappa^{-1} \sum_{l=1}^{p} \{\sum_{k<l} |L_{kl}|^2 + \sum_{k>l} |L_{lk}|^2\} \sigma(E_l)$, with $\sigma(E_l) = 1 + \frac{1}{2} \sum_{k \neq l} \dim \mathfrak{k}_{\frac{1}{2}(\alpha_l+\alpha_k)}$. Similarly $<X,X> = \omega([jX,X]) =$
$\omega([\sum_{l=1}^{p} s_l jE_l + \sum_{k<m} jL_{km}, \sum_{l=1}^{p} s_l E_l + \sum_{k<m} L_{km}]) = \sum_{l=1}^{p} s_l^2 \omega(E_l) + \sum_{k<m} \omega([jL_{km}, L_{km}])$

$= \kappa \sum_{l=1}^{p} s_l^2 + \sum_{k<m} |L_{km}|^2$, since $\omega(\mathfrak{k}_{\frac{1}{2}(\alpha_k+\alpha_m)}) = 0$. (Indeed $0 = \langle E_k, L_{km}\rangle = \omega([jE_k, L_{km}]) = \frac{1}{2}\omega(L_{km})$). So we have from $(X,X) = \mu^2\langle X,X\rangle$ that $\sigma(E_l) = \mu^2\kappa = \frac{d}{p}\ \forall l$, which gives us (D). *q.e.d.*

Remark. This is all we get; the rest of the identity $(X,X) = \mu^2\langle X,X\rangle$ is then satisfied if $\sigma(E_l) = \mu^2\kappa$.

We can now put everything together.

Theorem 1. Let $\mathcal{D}$ be an indecomposable, bounded, homogeneous domain, and let it be described by the normal j-algebra $(\mathfrak{g} = \mathfrak{r} + j\mathfrak{r} + \mathfrak{u}, j, \omega)$. Then $\mathcal{D}$ is quasi-symmetric if and only if $(\mathfrak{g}, j, \omega)$ satisfies the conditions (A),($\tilde{A}$),(B),(C),(D) and ($\tilde{D}$). If $\mathfrak{u} \neq (0)$, then we can skip (A), and if $\mathfrak{u} = (0)$, i.e. in the case of a tube domain, then ($\tilde{A}$) is void, of course, as is ($\tilde{D}$).

Proof. Only if: By Lemma 5 this follows from the earlier mentioned Vinberg-Koecher theorem ([11]), (21), Proposition 1, the above description of condition ($\tilde{D}$), and Lemma 6.
If: By the earlier considerations ($\tilde{A}$) and ($\tilde{D}$) imply (Q) (with respect to $\langle , \rangle$, the metric inherited from the Bergman metric), while Proposition 1 says that (A), (B) and (C) imply a compact Jordan structure on $\mathfrak{r}$. By [1], Ch. XI, there is the open convex cone $\tilde{\Omega}$, which is the component of the set of invertible elements of $\mathfrak{r}$ that contains E , and $\tilde{\Omega}$ is homogeneous with respect to $\Gamma(\mathfrak{r})^{\circ}$, the identity component of the structure group of $\mathfrak{r}$. By [1], p. 289, the Lie algebra of $\Gamma(\mathfrak{r})^{\circ}$ is $D(\mathfrak{r}) + A(\mathfrak{r})$, where $D(\mathfrak{r})$ is the set of derivations of $\mathfrak{r}$ and $A(\mathfrak{r})$ is the set of translations $A_X : Y \to XY$ of $\mathfrak{r}$ with $X \in \mathfrak{r}$. Also $[D(\mathfrak{r}),D(\mathfrak{r})] \subset D(\mathfrak{r})$, $[D(\mathfrak{r}),A(\mathfrak{r})] \subset A(\mathfrak{r})$ and $[A(\mathfrak{r}),A(\mathfrak{r})] \subset D(\mathfrak{r})$. By the construction of

the Jordan algebra we have $A_X = T_X = \frac{1}{2}\{\mathrm{adj}X+(\mathrm{adj}X)'\}$, and one checks easily, on the basis of the multiplication table for $\mathcal{l}$, that $D_X := \frac{1}{2}\{\mathrm{adj}X - (\mathrm{adj}X)'\} \in D(\mathcal{l})$ for all $X \in \mathcal{l}$. Hence $\mathrm{ad}\, jX \in D(\mathcal{l}) + A(\mathcal{l})$ for all $X \in \mathcal{l}$, and so $\Omega := \mathcal{G}_0 E \subset \Gamma(\mathcal{l})^\circ E = \tilde{\Omega}$. It remains to see that $\Omega = \tilde{\Omega}$ and that $\tilde{\Omega}$ is self-adjoint with respect to $\langle,\rangle$. Unfortunately [Lie algebra of $\mathcal{G}_0$, $D(\mathcal{l})$] is not zero in general, but we can argue as follows: By [9], p. 70, we have $\bar{\Omega} = \{Y \in \mathcal{l} \mid \omega(gY) \geq 0 \ \forall g \in \mathcal{G}_0\}$. (Actually Pyatetskii-Shapiro says $\Omega = \{Y \in \mathcal{l} \mid \omega(gY) > 0 \ \forall g \in \mathcal{G}_0\}$, but this author is only able to see the first version.) Now $\bar{\tilde{\Omega}} = \{X^2 \mid X \in \mathcal{l}\}$, by [1], p. 323, and if $g \in \mathcal{G}_0$, then $g \in \Gamma(\mathcal{l})^\circ$ by the above, so $gX^2 = Z^2$ some $Z \in \mathcal{l}$. If $Z = \sum_{l=1}^{p} s_l E_l + \sum_{k<m} L_{km}$, then $\omega(Z^2) = \sum_{l=1}^{p} s_l^2 \omega(E_l) + \frac{1}{2}\kappa^{-1} \sum_{k<m} |L_{km}|^2 (\omega(E_k)+\omega(E_m)) = \kappa \sum_{l=1}^{p} s_l^2 + \sum_{k<m} |L_{km}|^2 \geq 0$, since $\omega(\mathcal{k}_{\frac{1}{2}(\alpha_k+\alpha_m)}) = 0$ for $k<m$. So $X^2 \in \bar{\Omega}$, i.e. $\bar{\tilde{\Omega}} \subset \bar{\Omega}$. Since the cones are open and convex, we see that $\Omega = \tilde{\Omega}$. Further, by [1], p. 323, $\tilde{\Omega}$ is self-adjoint with respect to $(,)$.

Letting $\sigma(E_l) := 1 + \frac{1}{2} \sum_{k \neq l} \dim \mathcal{k}_{\frac{1}{2}(\alpha_k+\alpha_l)}$, we put, using condition (D), $\mu^2 = \sigma(E_l)\kappa^{-1}$, which is independent of l. By the calculation in Lemma 6, we have $(X,X) = \sum_{l=1}^{p} s_l^2 \sigma(E_l) + \frac{1}{2}\kappa^{-1} \sum_{l=1}^{p} \{\sum_{k<l} |L_{kl}|^2 + \sum_{k>l} |L_{lk}|^2\} \sigma(E_l)$ $= \mu^2 \{\kappa \sum_{l=1}^{p} s_l^2 + \sum_{k<m} |L_{km}|^2\} = \mu^2 \langle X,X\rangle$ for $X = \sum_{l=1}^{p} s_l E_l + \sum_{k<m} L_{km}$. Hence $(,)$ and $\langle,\rangle$ are proportional, and so $\Omega = \tilde{\Omega}$ is self-adjoint also with respect to $\langle,\rangle$. q.e.d.

<u>Remarks</u>: (1) In particular, for an indecomposable, symmetric domain, the condition (C) is satisfied. This was proved by d'Atri ([3]), also using j-algebras.

(2) The above proof rests, among other things, on Lemma 5, which, according to the remark following it, has not been checked for the exceptional Cayley cone.

Bibliography

[1] H. Braun und M. Koecher: Jordan Algebren. Grundlehren, Band 128, Springer Verlag, 1966.

[2] J.E. d'Atri: The curvature of homogeneous Siegel domains. To appear.

[3] J.E. d'Atri: Holomorphic Sectional Curvatures of Bounded Homogeneous Domains and Related Questions. To appear.

[4] S. Kaneyuki: On the automorphism groups of homogeneous bounded domains. J. Fac. Sci. Univ. Tokyo, 14 (1967), pp. 87-130.

[5] S. Kaneyuki: Homogeneous bounded domains and Siegel domains. Lecture Notes in Math., vol.241, Springer Verlag, 1971.

[6] W. Kaup, Y. Matsushima and T. Ochiai: On the automorphisms and equivalences of generalized Siegel domains. Amer. J. Math. 92 (1970), 475-498.

[7] J.L. Koszul: Sur la forme hermitienne canonique des espaces homogenes complexes. Canad. J. Math. 7 (1955), 562-576.

[8] S. Murakami: On automorphisms of Siegel domains. Lecture Notes in Math., vol.286, Springer Verlag, 1971.

[9] I.I. Pyatetskii-Shapiro: Automorphic Functions and the Geometry of Classical Domains. Gordon and Breach, New York-London-Paris, 1969.

[10] I. Satake: On classification of quasi-symmetric domains. Nagoya Math. J. 62 (1976), pp. 1-12.

[11] E.B. Vinberg: Homogeneous cones. Dokl. Akad. Nauk SSSR 133 (1960), pp. 9-12 ; = Soviet Math. Dokl. 1 (1960), pp. 787-790.

[12] R. Zelow (Lundquist): Holomorphic Sectional Curvature of Quasi-Symmetric Domains. To appear in Proceedings of the Am. Math. Soc.

[13] R. Zelow (Lundquist): **Curvature** of Quasi-Symmetric Domains. To appear in Journal of Differential Geometry.